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Uniqueness of extensions of homogeneous polynomials on c_0-sum of Hilbert spaces

by M.L. Lourenço and L. Pellegrini

Departamento de Matemática, Universidade de São Paulo, CP 66281, CEP: 05315-970 São Paulo, Brazil

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ABSTRACT

We study the uniqueness of norm-preserving extension of n-homogeneous polynomials on X, where X is a c_0-sum of Hilbert spaces. We show that there exists a unique norm-preserving extension for norm-attaining 2-homogeneous polynomials on X to X'', but this result fails for homogeneous polynomials of degree greater than 2.

1. INTRODUCTION

There is no Hahn–Banach theorem for n-homogeneous polynomials on Banach space. Even when a homogeneous polynomial can be extended to a larger subspace, it can happen that the norm cannot be preserved. For example, let $E = (\mathbb{C}^3, \|\cdot\|_\infty)$ and consider the subspace $F = \{(x, y, z) \in E\colon x + y + z = 0\}$. The 2-homogeneous polynomial on F defined by $P(\alpha + \beta, -\alpha, -\beta) = \alpha^2 + \alpha\beta + \beta^2$ has norm 1, but each extension of P to E has norm greater than 1 (see [1]). The Aron–Berner extension process for homogeneous polynomials on a Banach spaces [1] avoids these obstacles by requiring that the extension be defined only for a very restricted class of spaces containing E, essentially the bidual E''. More than ten years latter, Davie and Gamelin [4] proved that this canonical extension constructed in [1] is norm-preserving. Recently, several authors [2,3,6] have studied the question when the norm-preserving extension of n-homogeneous polynomials from some classical

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E-mails: mllouren@ime.usp.br (M.L. Lourenço), leonardo@ime.usp.br (L. Pellegrini).

Banach space E to its bidual E'' is unique. In this paper, we study the analogous problem in spaces that are c_0-sum of Hilbert spaces. We will show in Section 2 that there exists a unique norm-preserving extension for norm-attaining 2-homogeneous polynomials on this kind of spaces. In Section 3, we will show that this result cannot be extended to n-homogeneous polynomials if $n \geqslant 3$.

We refer to [5] and [7] for notation and results regarding homogeneous polynomials.

Let I be an infinite set of indices and $(H_i)_{i\in I}$ a family of complex Hilbert spaces parametrized by I. Let $X = c_0^H$ denote the Banach space of all $x = (x_i)_{i\in I}$ such that $x_i \in H_i$ for every $i \in I$ and $(\|x_i\|)_{i\in I} \in c_0(I)$, provided with the norm $\|x\| = \sup_{i\in I} \|x_i\|$. We say that X is a c_0-sum of the Hilbert spaces H_i. The strong bidual X'' of X is canonically isomorphic to l_∞^H, the space of all $x = (x_i)_{i\in I}$ such that $x_i \in H_i$ for every $i \in I$ and $(\|x_i\|)_{i\in I} \in l_\infty(I)$ provided with the same supremum norm.

2. EXTENSION OF POLYNOMIALS

Let E be a complex Banach space, and let B_E be the unit ball of E. We will denote by $P(^m E)$ the Banach space of all continuous m-homogeneous polynomials on E with the norm $\|P\| = \sup_{x\in B_E} |P(x)|$. A polynomial $P \in P(^m E)$ is *norm-attaining* if there exists $x \in B_E$ such that $\|P\| = P(x)$.

Our interest here will be focused on the following: Given a norm-attaining polynomial P in $P(^2X)$, there is a unique norm-preserving extension $\tilde{P}$ to $P(^2X'')$.

Theorem 2.1. *Let $P \in P(^2X'')$, $P \neq 0$, and let $x^0 \in X$ such that $\|x^0\| \leqslant 1$ and $|P(x)| \leqslant |P(x^0)|$ for all $x \in X''$ such that $\|x\| \leqslant 1$. Then $\|x^0\| = 1$, the set J of all $j \in I$ such that $\|x_j^0\| = 1$ is finite, and $P(x)$ only depends on the x_j with $j \in J$.*

Proof. Let P be a 2-homogeneous polynomial on X'' satisfying the hypothesis. So, we have $|P(x^0)| > 0$ hence $\|x^0\| > 0$, and $|P(tx^0)| = t^2|P(x^0)| > |P(x^0)|$ if $t = 1/\|x^0\| > 1$, and this proves that $\sup_{i\in I} \|x_i^0\| = \|x^0\| = 1$.

Without loss of generality we may assume in the sequel that $P(x^0) = 1$. Because $(\|x_i^0\|)_{i\in I} \in c_0(I)$, for each $\varepsilon > 0$ the set I_ε of all $i \in I$ such that $\|x_i^0\| \geqslant \varepsilon$ is finite. So, chosen $\varepsilon = 1$ we have that J is finite and not empty. It also follows that $\delta := \sup_{i\in I\setminus J} \|x_i^0\| < 1$.

Let $Y = \{y \in X'':\ y_j = 0, \forall j \in J\}$. Then Y is a closed linear subspace of X''. The set $Z = \{z \in X'':\ z_i = 0, \forall i \in I \setminus J\}$, which is canonically identified with the Cartesian product of the finitely many Hilbert spaces H_j, $j \in J$, is a closed linear complement of Y in X''. If $y \in Y$, $\|y\| \leqslant 1$, $t \in \mathbb{C}$, $|t| \leqslant 1 - \delta$, where $1 - \delta > 0$, then $\|x^0 + ty\| \leqslant 1$, hence

$$1 \geqslant \left|P\left(x^0 + ty\right)\right| = \left|1 + 2t\check{P}\left(x^0, y\right) + t^2 P(y)\right|.$$

If $\check{P}(x^0, y) \neq 0$, then with the argument of t equal to minus the argument of $\check{P}(x^0, y)$, the right hand side is of order $1 + |t||\check{P}(x^0, y)|$ for small positive $|t|$, which leads to a contradiction. Thus $\check{P}(x^0, y) = 0$. If $P(y) \neq 0$, then with the argument of

t equal to minus one half of the argument of $P(y)$ the right hand side is of order $1+|t|^2|P(y)|$ for small positive $|t|$, again leading to a contradiction. Therefore $P(y)=0$ for all $y\in Y$. In turn this implies that $\check{P}(u,y)=0$ for all $u,y\in Y$.

In the sequel of this proof, we identify $u\in H_i$ with the element $x\in X''$ such that $x_i=u$ and $x_h=0$ when $h\in I\setminus\{i\}$. Let $j\in J$ and consider $x(s)=x^0+(e^{is}-1)x_j^0$, $s\in\mathbb{R}$. Then $\|x(s)_i\|=\|x_i^0\|$ for every $i\in I$, and therefore, if again $y\in Y$, $\|y\|\leqslant 1$, $t\in\mathbb{C}$, $|t|\leqslant 1-\delta$, we have $\|x(s)+ty\|\leqslant 1$, hence $|P(x(s)+ty)|\leqslant 1$, and so

$$\begin{aligned}
1&\geqslant \left|P(x(s))+2t\check{P}(x(s),y)\right|\\
&=\left|P(x^0)+2(e^{is}-1)\check{P}(x^0,x_j^0)+(e^{is}-1)^2P(x_j^0)+2t(e^{is}-1)\check{P}(x_j^0,y)\right|\\
&=\left|1+2(e^{is}-1)\big(\check{P}(x^0,x_j^0)+t\check{P}(x_j^0,y)\big)+(e^{is}-1)^2P(x_j^0)\right|.
\end{aligned}$$

For small $|s|$, $e^{is}-1=is+\mathcal{O}(s^2)$, and we get that

$$\left|1+2is\big(\check{P}(x^0,x_j^0)+t\check{P}(x_j^0,y)\big)+\mathcal{O}(s^2)\right|\leqslant 1.$$

This inequality would lead to a contradiction if $\check{P}(x^0,x_j^0)+t\check{P}(x_j^0,y)\notin\mathbb{R}$, hence $\check{P}(x^0,x_j^0)+t\check{P}(x_j^0,y)\in\mathbb{R}$. The fact that this holds for every $t\in\mathbb{C}$ such that $|t|\leqslant 1-\delta$ now implies that $\check{P}(x_j^0,y)=0$ and $P(x^0,x_j^0)\in\mathbb{R}$. The fact that this holds for every $y\in Y$ such that $\|y\|\leqslant 1$ implies that $\check{P}(x_j^0,y)=0$ for every $y\in Y$.

Let $j\in J$ and let $u\in H_j$ such that $\|u\|=1$ and $u\in\{x_j^0\}^\perp$. For $s\in\mathbb{R}$, consider this time $x(s)=x^0+(1+s^2)^{-1/2}(x_j^0+su)-x_j^0$. So, if $i\neq j$, $x(s)_i=x_i^0$, and $x(s)_j=(1+s^2)^{-1/2}x_j^0+(1+s^2)^{-1/2}su$, hence

$$\|x(s)_j\|^2=\left(1+s^2\right)^{-1}\left(\|x_j^0\|^2+s^2\|u\|^2\right)=\left(1+s^2\right)^{-1}\left(1+s^2\right)=1.$$

Then again $\|x(s)_i\|=\|x_i^0\|$ for every $i\in I$, and if $y\in Y$, $\|y\|\leqslant 1$, $t\in\mathbb{C}$, $|t|\leqslant 1-\delta$, $\|x(s)+ty\|\leqslant 1$, hence

$$1\geqslant\left|P\big(x(s)+ty\big)\right|=\left|1+2s\left(1+s^2\right)^{-1/2}\big(\check{P}(x^0,u)+t\check{P}(u,y)\big)+\mathcal{O}(s^2)\right|$$

for $s\to 0$. As above this implies that $\check{P}(x^0,u)+t\check{P}(u,y)\in i\mathbb{R}$ and therefore $\check{P}(u,y)=0$ for all $y\in Y$ for all u in the orthogonal complement of x_j^0 in H_j.

Now, let $y\in Y$. Since H_j is equal to the direct sum of $\mathbb{C}x_j^0$ and its orthogonal complement in H_j and $z\mapsto\check{P}(z,y)$ is linear, it follows that $\check{P}(z,y)=0$ for all $z\in H_j$ and for all $j\in J$. Again using the linearity, we obtain that $\check{P}(z,y)=0$ for all $z\in Z$ and $y\in Y$. In combination with $P(y)=0$ for all $y\in Y$, this shows that $P(y+z)=P(z)$ for all $y\in Y$, $z\in Z$, which is the conclusion of the theorem. □

Remarks 2.1. (a) In the proof of Theorem 2.1 is essential that X is a c_0-sum of complex Hilbert spaces. Indeed, let I be the set of all nonnegative integers, $H_0=\mathbb{C}^2$

provided with the norm $\|x_0\| = |x_{0,1}| + |x_{0,2}|$ and $H_i = \mathbb{C}$ for all $i > 0$. Define the 2-homogeneous polynomial $P : X'' \to \mathbb{C}$ by

$$P(x) = x_{0,1}^2 + x_{0,2} \sum_{i=1}^{\infty} 2^{-i} x_i.$$

If $\|x\| \leqslant 1$, then $|x_{0,1}| + |x_{0,2}| \leqslant 1$ and $|x_i| \leqslant 1$ for all $i > 0$, hence $|P(x)| \leqslant |x_{0,1}|^2 + |x_{0,2}| \leqslant |x_{0,1}|^2 + 1 - |x_{0,1}|$. Because the maximum of a convex function on an interval is attained at the end points, we have $|P(x)| \leqslant 1$ for all x such that $\|x\| \leqslant 1$. On the other hand $P(x_0) = 1$ when $x_0 = (1, 0)$ and $x_i = 0$ for all $i > 0$. That is, $\|P\|$ attains its maximum in the unit ball in X, but $P(x)$ depends on infinitely many variables.

(b) If in (a) we take the norm $\|x_0\| = \max(|x_{0,1}|, |x_{0,2}|)$ in H_0, then Theorem 2.1 holds although H_0 is not a Hilbert space, because in this case X is isomorphic to c_0.

As a consequence of the Theorem 2.1, we have the following uniqueness of the norm-preserving extension for the norm-attaining 2-homogeneous polynomials.

Corollary 2.2. *Let P be a norm-attaining 2-homogeneous polynomial on X. Then, there exists a unique norm-preserving extension of P to X''.*

Proof. Let $\bar{P}$ be a norm-preserving extension of P to X''. Since P is norm-attaining, $\bar{P}$ attains its norm in a point $x^0 \in B_X$. By Theorem 2.1 $\bar{P}(x)$ only depends on the x_j with $j \in J$, where J is a finite set. Another norm-preserving extension of P coincides with $\bar{P}$, since X contains $Z = \{y \in X'' \colon y_j = 0, \forall j \in I \setminus J\}$. $\square$

The converse of the Corollary 2.2 is not true. In fact, when $X = c_0$, the 2-homogeneous polynomial $P(x) = \sum_{i \in \mathbb{N}} \frac{1}{2^i} x_i$, $x = (x_1, x_2, \ldots) \in c_0$ is not norm-attaining, but it has a unique norm-preserving extension to l_∞ (see [2]).

3. GENERALITIES

In this section we will prove that the Corollary 2.2 is not true for homogeneous polynomials with degree greater than 2. In other words, for each $n \geqslant 3$, there exists a n-homogeneous polynomial that attains its norm, but it has at least two norm-preserving extensions. First, we need the following lemma, which will be useful here:

Lemma 3.1. *Let $n \in \mathbb{N}$, $n \geqslant 3$. Then*

$$|\lambda + \mu|^n + 2|\lambda - \mu|^{n-1} \leqslant 2^n,$$

$\forall \lambda, \mu \in \mathbb{C}$, *with* $|\lambda| \leqslant 1$ *and* $|\mu| \leqslant 1$.

Proof. Note that $|\lambda + \mu|^n + 2|\lambda - \mu|^{n-1}$ is equal to $f(s) := (t + s)^{n/2} + 2(t - s)^{(n-1)/2}$, where $t := |\lambda|^2 + |\mu|^2$ and $s := \lambda\bar{\mu} + \bar{\lambda}\mu$. Observe that $0 \leqslant t \leqslant 2$ and

$-t \leqslant s \leqslant t$. Now $n \geqslant 3$ implies that $f''(s) > 0$ when $-t < s < t$, which implies that f is convex on $[-t, t]$, and therefore

$$f(s) \leqslant \max\bigl(f(-t), f(t)\bigr) = \max\bigl(2(2t)^{(n-1)/2}, (2t)^{n/2}\bigr)$$
$$\leqslant \max\bigl(2\,(4)^{(n-1)/2}, 4^{n/2}\bigr) = 2^n,$$

which proves the lemma. □

For each $n \geqslant 3$, we will use the lemma to find a n-homogeneous polynomial P that attains its norm which has, at least, two norm-preserving extensions. As the set of all norm-preserving extensions of P is a convex set then P has infinite many extensions.

Proposition 3.2. *Let X be a c_0-sum of complex Hilbert spaces. If $n \geqslant 3$, then there exists a norm-attaining n-homogeneous polynomial on X which does not have unique norm-preserving extension to X''.*

Proof. For each H_i, $i \in I$ we take a orthonormal (Hilbert) basis $\{e_{i,j}\colon j \in \kappa_i\}$. Fix $i_1, i_2 \in I$ and consider $P \in P({}^nX)$, $n \geqslant 3$, defined by

$$P(x) = (x_{i_1,1} + x_{i_2,1})^n, \quad x = (x_i)_{i \in I},\ x_i = \sum_{j \in \kappa} x_{i,j} e_{i,j}.$$

It is easy to see that $\|P\| = 2^n$. We claim that P does not have unique norm-preserving extension to X''.

Note first that $P_1 \in P({}^nX'')$ defined by

$$P_1(x) = (x_{i_1,1} + x_{i_2,1})^n, \quad x = (x_i)_{i \in I},\ x_i = \sum_{j \in \kappa} x_{i,j} e_{i,j}$$

is a norm-preserving extension of P.

Next, taking $x^0 \in X''$ with $x^0_{i_2} = -e_{i_2,1}$ and $x^0_i = e_{i,1}$ for $i \neq i_2$, we have that $d(x^0, X) = 1$. So, by the Hahn–Banach theorem there exists a $\varphi \in X'''$ such that $\|\varphi\| = 1$, $\varphi|_X \equiv 0$ and $\varphi(x^0) = 1$. In this case, we define the 2-homogeneous polynomial P_2 on X'' by

$$P_2(x) = (x_{i_1,1} + x_{i_2,1})^n + 2(x_{i_1,1} - x_{i_2,1})^{n-1} \cdot \varphi(x).$$

Since $\varphi|_X \equiv 0$, $P_2|_X = P$. We have that $P_1 \neq P_2$ in X'' because $P_1(x^0) = 0$ and $P_2(x^0) = 2^n$.

Now, the proof will be complete once we show that $\|P_2\| \leqslant 2^n$.

Firstly observe that for λ and μ complex numbers with $|\lambda| \leqslant 1$, $|\mu| \leqslant 1$ and for $z \in X''$ with $\|z\| \leqslant 1$ and $z_{i_1,1} = z_{i_2,1} = 0$ we get

$$\begin{aligned}|P_2(\lambda\tilde{e}_{i_1,1} + \mu\tilde{e}_{i_2,1} + z)| &= \bigl|(\lambda + \mu)^n + 2(\lambda - \mu)^{n-1} \cdot \varphi(z)\bigr| \\ &\leqslant |(\lambda + \mu)|^n + 2|(\lambda - \mu)|^{n-1}.\end{aligned}$$

For the equality above we used that φ vanishes in X and for the inequality that the norm of φ is 1. Hence, for each fixed $z \in B_{X''}$ as above we have

$$\sup_{|\lambda|,|\mu|\leqslant 1} |P_2(\lambda e_{i_1,1} + \mu e_{i_2,1} + z)| \leqslant \sup_{|\lambda|,|\mu|\leqslant 1} \left(|(\lambda+\mu)|^n + 2|(\lambda-\mu)|^{n-1}\right) \leqslant 2^n,$$

by Lemma 3.1. In particular, $|P_2(x)| \leqslant 2^n$, for $x \in X''$ with $\|x\| \leqslant 1$. $\square$

We observe that when X is a c_0-sum of *real* Hilbert spaces even the norm-attaining 2-homogeneous polynomials could not have a unique norm-preserving extension. Indeed, $X = c_0$, and $n \geqslant 2$ the polynomial $P(x) = x_1^n$, $x = (x_1, x_2, \ldots) \in c_0$ has at least two extensions to l_∞ (see [2]).

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